

## The Free Particle:

Consider  $V(x) = 0$ . No PE, just a free particle.  
 We talked about this at the start, (de Broglie's ideas)  
 Now we can tackle it with the Schrodinger Eq'n.  
 There is some funny business here, but clearly this too is  
 an important physics case, we need to describe quantum  
 systems that aren't bound!

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$$-\frac{\hbar^2}{2m} u''(x) = E u(x) \quad \text{that looks easy enough!}$$

$$\text{Define } K = \frac{\sqrt{2mE}}{\hbar}, \text{ so } u''(x) = -K^2 u(x).$$

I can solve that 2<sup>nd</sup> order ODE by inspection..

$$u_k(x) = A e^{iKx} + B e^{-iKx} \quad \left( \text{The label "K" identifies } u, \text{ different K's } \Rightarrow \text{different } u\text{'s} \right)$$

There is no boundary condition, so  $K$  (thus  $E$ ) is  
not quantized. Free particles can have any energy!

$$\text{And } \Psi_k(x, t) = (A e^{iKx} + B e^{-iKx}) e^{-iEt/\hbar}$$

$$\left( \text{Let's define } \omega = E/\hbar, \text{ as usual} \right) \quad \left( \text{with } E = \hbar^2 K^2 / 2m \right)$$

Rewriting:  $\Psi_K(x,t) = \underbrace{A e^{iK(x - \frac{\omega}{K}t)}}_{\text{this is a fn of } x - vt} + \underbrace{B e^{-iK(x + \frac{\omega}{K}t)}}_{\text{fn of } x + vt}$

this is a fn of  $x - vt$

right-moving wave!

speed  $v = \omega/K$  (called "phase velocity")

left moving wave

speed  $\omega/K$

Here,  $v = \frac{\omega}{K} = \frac{E/\hbar}{K} = \frac{\hbar^2 K^2 / 2m}{\hbar K} = \frac{\hbar K}{2m}$  //

what did we expect? I'm thinking  $p = \hbar K^*$  so  $V = \frac{p}{m} = \frac{\hbar K}{m}$

what's with that funny factor of 2, the  $\hbar$ ?

Also, the 2 terms can be written like this:

$$\Psi_K(x,t) = A e^{i(Kx - \frac{\hbar K^2}{2m}t)}$$

where we let  $K$  be +  
or -,  $K > 0 \Rightarrow$  right moving  
 $K < 0 \Rightarrow$  left "

For this function,

$$* \hat{p} \Psi_K = \frac{\hbar}{i} \frac{\partial}{\partial x} \Psi_K = \hbar K \Psi_K$$

so this  $\Psi_K$  is an eigenfunction of  $\hat{p}$ , with eigenvalue  $\hbar K$

Just exactly as our "de Broglie intuition" says

(The  $\Psi_K$  at the top of the page, (with  $K > 0$  only) is a mix....)

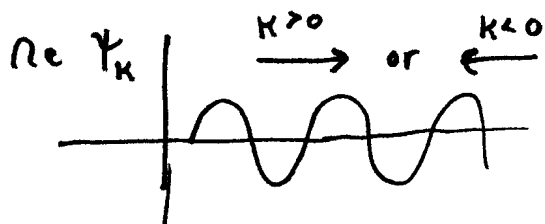
Summary

So we have sol'ns  $\Psi_K(x,t) = A e^{i(Kx - \omega t)}$

where  $K = \pm \frac{\sqrt{2mE}}{\hbar}$ , (2 values of  $K$  for each  $E$ )

These waves have  $\lambda = \frac{2\pi}{|K|}$  ( $\Psi(x \pm \lambda) = \Psi(x)$ )

speed (phase velocity) =  $\frac{\hbar|K|}{2m} = \frac{h}{2m\lambda}$  (small  $\lambda \Rightarrow$  faster speed)



$\hookrightarrow$  funny "2" here,  
classically expected  $v = \frac{p}{m} = \frac{\hbar K}{m}$   
 $= \sqrt{2E/m}$

Funny  $v$  is one issue we need to understand.

Here's another, this  $\Psi$  is not normalizable!

$$\int_{-\infty}^{\infty} |\Psi_K(x,t)|^2 dx = A^2 \int_{-\infty}^{\infty} 1 \cdot dx \leftarrow \text{yikes!}$$

Conclusion: this  $\Psi_K$  is not representing a physical object.

There are no free (quantum) particles with definite energy.

But... not to worry, these  $\Psi_K$ 's are useful, even

essential! We can form linear combos that are physical!

Since  $K$  is a continuous variable, combining  $\Psi_K$ 's requires not a sum over  $K$ , but an integral over  $K$ !

Consider

$$\Psi_{\text{general}}^{\text{free}}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(K) \Psi_K(x,t) dK$$

Stuck this in for later convenience

This plays the role of  $C_n$  before,  $n$ 's a function of  $K$

Sum over all  $K$ 's, + or -.

(This is the continuous analogue of square well)

$$\Psi_{\text{general}}^{\text{square well}}(x,t) = \sum_n C_n \Psi_n^{\text{square well}}(x,t), \text{ replace } C_n \rightarrow \phi(K) dK$$

• This  $\Psi$  has many  $K$ 's (momenta, energies) superposed

• If I give you  $\Psi_{\text{general}}(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(K) \Psi_K(x,0) dK$

then you can figure out  $\phi(K)$  (like Fourier's trick!)

+ given  $\phi(K)$ , you then know, from eq'n at top of page,  $\Psi$  at all future times.

So we need to review "Fourier's trick" for continuous  $K$ !

Note  $\Psi_K(x,0) = e^{iKx}$  ← simple enough ...

Suppose I give you  $f(x)$ , and ask "what  $\phi(k)$  is needed" so that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} dx$$

[This is our problem: Given  $\Psi(x, t=0)$ , find  $\phi(k)$ .]

Because once you've got it, you know  $\Psi$  at all times

the answer is Plancherel's theorem, also known as the "Fourier transform"

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

Note this sign

[Recall that if  $f(x) = \sum_n c_n \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$ , then

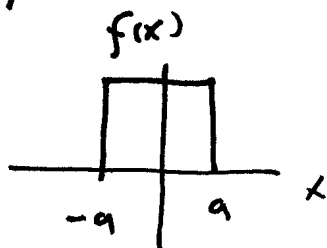
Fourier's trick gave  $c_n = \int f(x) \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} dx$

It's rather similar, the coefficients are integrals of the desired function with our "orthogonal basis" functions.

- $\phi(k)$  is the "Fourier transform" of  $f(x)$  here.
- If  $f(x)$  is normalizable to start with,  $\Psi(x, t)$  will be too.  
 $\Psi(x, t=0)$  and so will  $\phi(k)$ .
- You can "build" almost ANY FUNCTION  $f(x)$  you want !!

Digression - some examples of Fourier transforms.

Griffiths (p. 62) has

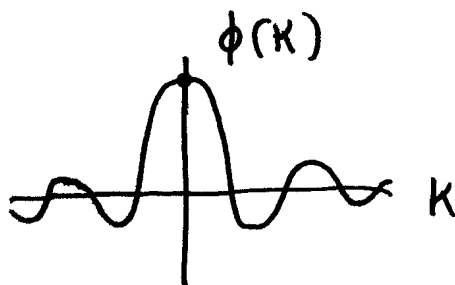


"width" is  $\approx 2a$ .

then  $\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$  is quite easy, + you

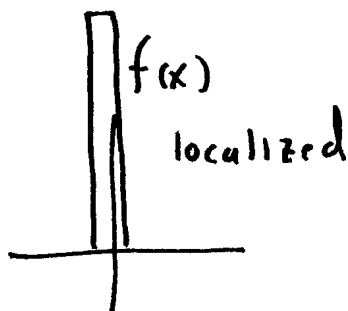
get

$$\phi(k) = \frac{1}{\sqrt{\pi a}} \frac{\sin ka}{k}$$

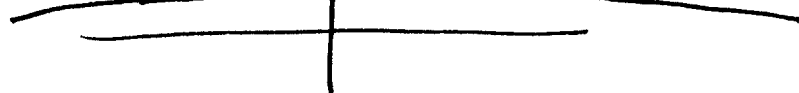


$\pi/a$  is "width", roughly

If  $a \rightarrow 0$ ,

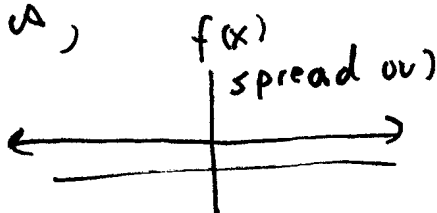


$\phi(k) \sim \text{constant}$   
spread out

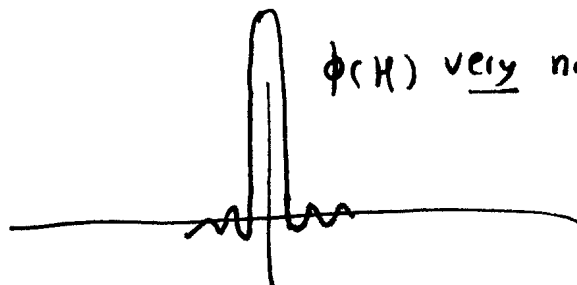


Need many momenta to build a narrow wave packet

If  $a \rightarrow \infty$ ,



$\phi(k)$  very narrow

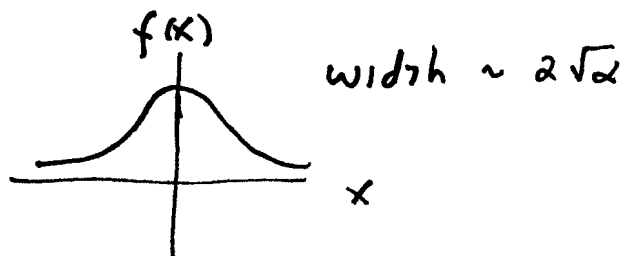


one (sharp) momentum  $\Rightarrow$  broad wave packet, where is it?

Digression continued:

Let's do another:

$$f(x) = A e^{-x^2/4\alpha}$$



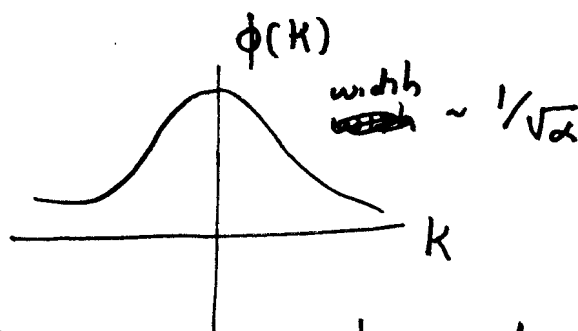
Here also,  $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$  can be done analytically,

$$\phi(k) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{4\alpha} + ikx\right)} dx = \frac{A}{\sqrt{2\pi}} e^{-k^2\alpha} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{2\sqrt{\alpha}} - ik\sqrt{\alpha}\right)^2} dx$$

"completing the square"

Letting  $x' = \frac{x}{2\sqrt{\alpha}} - ik\sqrt{\alpha}$ ;  $dx' = \frac{dx}{2\sqrt{\alpha}}$ ;  $\int_{-\infty}^{\infty} e^{-x'^2} dx' = \sqrt{\pi}$

$$\phi(k) = \frac{2\sqrt{\alpha} A}{\sqrt{2\pi}} e^{-k^2\alpha}$$



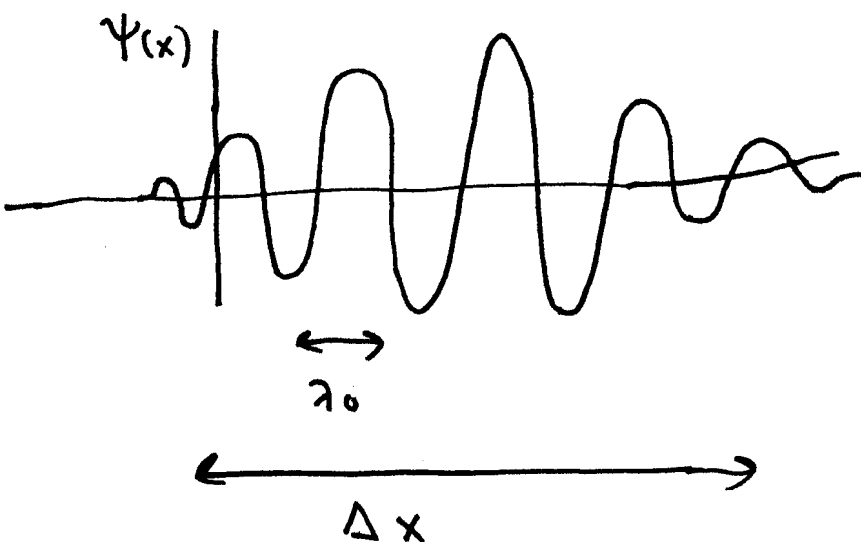
once again, narrow in  $f \Rightarrow$  wide in  $\phi$ , and vice versa.

• If  $f$  is centered around  $x_0$ ,  $f(x) = A e^{-(x-x_0)^2/4\alpha}$ ,

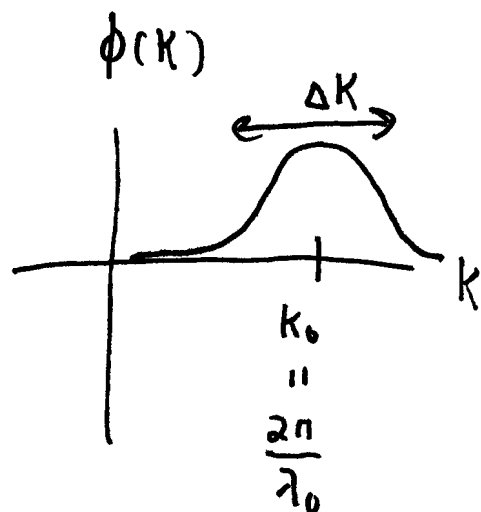
work it out... This will add a phase  $e^{-ikx_0}$  to  $\phi(k)$

so phase of  $\phi$  does carry some important information (but not about momentum itself!) (what does phase  $e^{ik_0 x}$  in  $f(x)$  do? ...)

So instead of  $\Psi(x) = \text{Pure } e^{ikx}$ , we will start with a more physical  $\Psi$ , a "wave packet"



which has a Fourier transform:



Any  $\Psi(x)$  can be "built" like this, the math of Four. transforms yields the following, in general:

- $\phi(k)$  may very well turn out complex, (but might also be real)
- If  $\Psi(x)$  is "mostly sinusoidal" with wavelength  $\sim \lambda_0$  (like above)

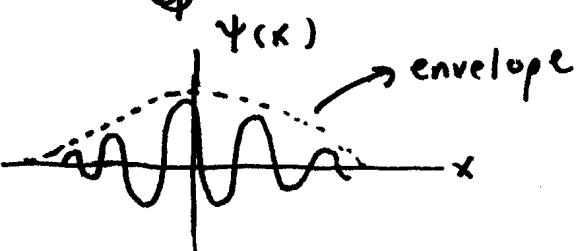
then  $\phi(k)$  will be centered around  $k = 2\pi/\lambda_0$

- If  $\Psi(x)$  is localized with size  $\Delta x$ ,  
then  $\phi(k)$  is " " "  $\Delta k \approx \frac{1}{\Delta x}$ .

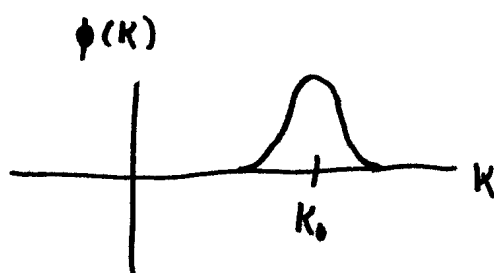
- Given  $\Psi(x, t=0) \Rightarrow \phi(k) \Rightarrow \Psi(x, t)$  is determined for all times



claim: If start with a simple wave packet,



$\Rightarrow$



As time goes by, the ripples in the  $\psi(x)$  wave packet will move with phase velocity  $\omega/k$ , but the envelope itself (which we interpret as "where the particle is located"!) moves with a different velocity, "group velocity"  $= \frac{d\omega}{dk}$

In this case,  $\omega = \frac{E}{\hbar} = \frac{\hbar}{2m} k^2$ , so  $\frac{d\omega}{dk} = \frac{\hbar k}{m}$ ,

no funny factor of 2! This resolves both our "free particle issues", 1) the packet moves with  $v_{\text{classical}} = \hbar k/m$

2) the  $\Psi(x,t)$  is perfectly normalizable.

• And something else happens: different  $k$ 's move with different

speeds, so the ripples tend to spread out the packet as

time goes by, so  $\rightarrow$  higher  $k$ 's move up faster

so  $\Delta x$  grows with time!

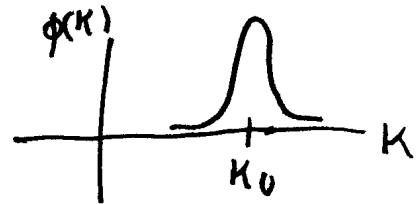
# SJP 3220 QM Notes 2.36

Casual proof of claim:

$$\Psi_{\text{general}}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \omega t)} dk$$

Recall  $\omega$  is itself dependent on  $k$ , ( $\omega = \frac{\hbar^2 k^2}{2m}$  in this case)

Let's consider a wave packet where  $\phi(k)$  is peaked near  $k_0$ .



So we have a "reasonably well-defined" momentum, (this is what you think of when you have a real particle like an electron in a beam ...) (If you had many  $k$ 's, each  $k$  travels at different speeds  $\Rightarrow$  packet spreads, not really a "particle" ...)

So this is a special case, but a common/practical one...

So if  $k$ 's are localized, consider Taylor expanding  $\omega(k)$

$$\omega(k) = \omega(k_0) + \omega'(k_0)(k - k_0) + \dots$$

this should be a fine approx for  $\omega(k)$  in that integral we need to do at the top of the page!

Let's define  $k' \equiv k - k_0$  as a better integration variable,

$$\text{so } dk' = dk, \text{ but } \omega \approx \omega(k_0) + \omega'(k_0)k' + \dots$$

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$$\begin{aligned}\Psi_{\text{gen}}(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k'+k_0) e^{i(k'+k_0)x} e^{-i\omega(k_0)t} e^{-i\omega'(k_0)k't} dk' \\ &= e^{i k_0 x} e^{-i\omega(k_0)t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k'+k_0) e^{i k' x} e^{-i\omega'(k_0) \cdot k' \cdot t} dk' \\ &= e^{i(k_0 x - \omega_0 t)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k_0+k') e^{i k' (x - \omega'_0 t)} dk'\end{aligned}$$

these are the ripples,  
a function of

$$x - \frac{\omega_0}{k_0} t$$

Travel with  $v_{\text{phase}} = \frac{\omega_0}{k_0}$

This is a function of  $(x - \omega'_0 t)$

This is the time-evolving envelope,  
which travels at speed  $\omega'_0$

that's the group velocity,  $\omega'_0 = \left. \frac{d\omega}{dk} \right|_{k_0}$

So the Quantum free wave packet moves with  $v = \omega'_0 \approx \frac{\hbar k_0}{m}$

Just as you'd expect classically.

the ripples travel at  $v_{\text{phase}}$ , but  $|\Psi|^2 \Rightarrow$  don't care about  
the ripples, really

wave packets do "spread out" in time, we could compute  
this (turns out it spreads faster if starts out narrower!)

We're now armed to solve a variety of problems:

Given  $V(x)$ , find  $\Psi(x,t)$ .  
(and  $\Psi(t=0)$ )

→ Bound states will generate discrete  $E_n$ 's +  $u_n$ 's

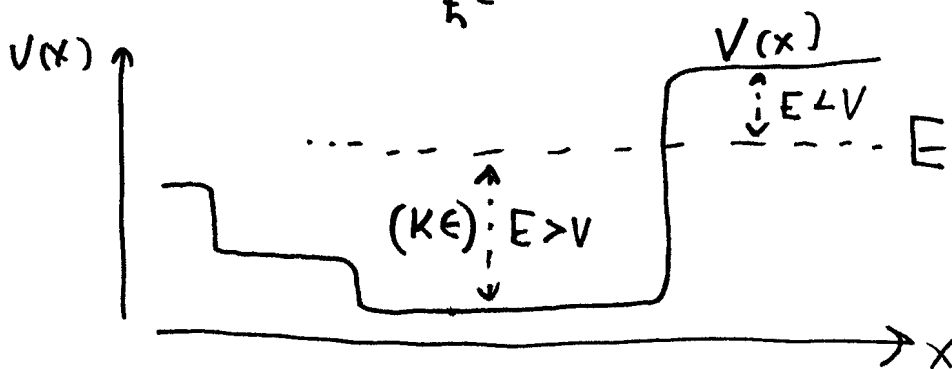
→ Free particles will be "scattering states". We'll talk more about this soon. ~~Let's~~ let's go back & tackle a few more ~~simple~~ <sup>simple</sup> ~~problems~~ problems, to get familiar with  $\Psi$ 's...

TISE:  $\hat{H} u_n(x) = E_n u_n(x)$

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} u(x) + V(x) u(x) = E u(x)$$

$$u''(x) = -\frac{2m}{\hbar^2} (E - V(x)) u(x)$$

← so  $u$  and  $u'$  need to be continuous, at least if  $V$  is finite.



If  $E > V$ ,  $u'' = -(K^2)u$

Classically,  $KE > 0$

$$K^2 \equiv \frac{2m(E-V)}{\hbar^2} \quad \text{or} \quad E-V = \frac{\hbar^2 K^2}{2m} \stackrel{\text{de Broglie}}{=} \frac{p^2}{2m} \quad \downarrow \quad \text{as usual!}$$

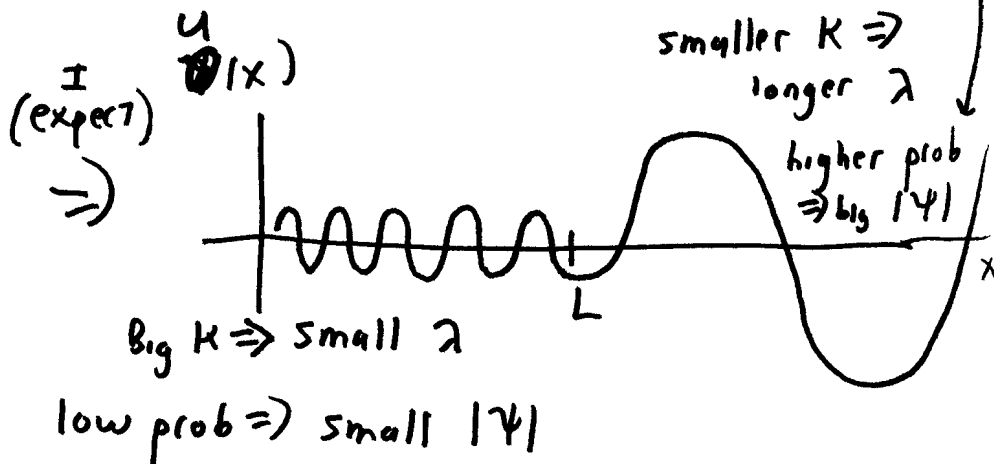
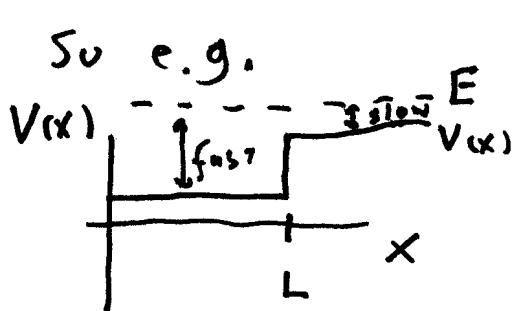
$u'' = -K^2 u \Rightarrow$  curvature is towards axis.

this is a wiggly function!

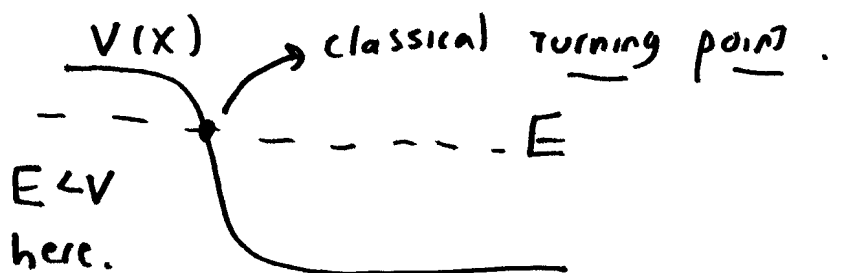
If  $K^2$  is constant,  $u = A \sin Kx + B \cos Kx$   
or  $A'e^{iKx} + B'e^{-iKx}$

Big KE  $\Rightarrow$  Big  $K^2 \Rightarrow$  more wiggly (smaller  $\lambda$ )

Classically Big KE  $\Rightarrow$  faster speed  $\Rightarrow$  less likely to be found there. This is very different, it corresponds in QM not to  $\lambda$ , but Amplitude (usually, there are exceptions)



What if  $E < V$ ? Classically this never happens (negative KE?!)



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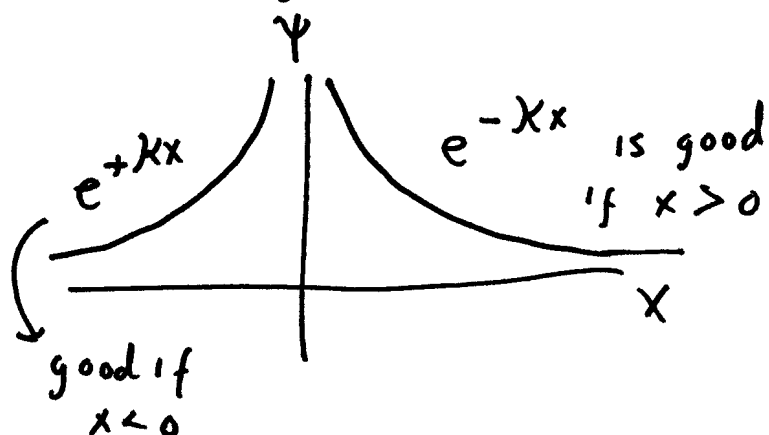
In Q.M,  $u''(x) = -\frac{2m}{\hbar^2} (E - V(x)) u(x)$

If  $E < V$ , define  $K = +\frac{\sqrt{2m(V-E)}}{\hbar}$ , so  $u'' = +K^2 u$

This is not wiggly, curvature is away from axis

If  $K^2 = \text{constant}$  (constant  $V$  in a region)

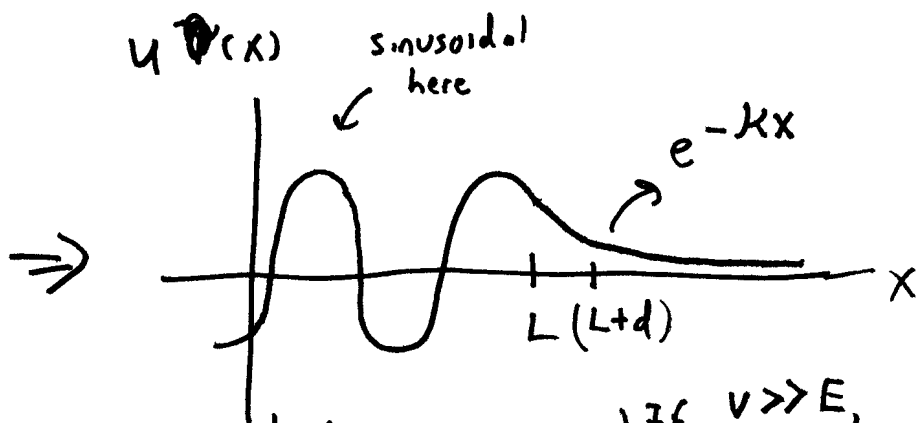
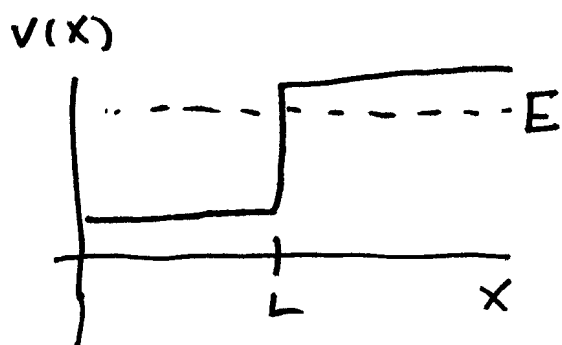
$$u(x) = Ce^{-Kx} + De^{+Kx}$$



The other sigs, e.g.  $e^{+Kx}$  for  $x > 0$  blow up @  $\infty$ , no good!

So in "classically forbidden" regions, can have  $u(x) \neq 0$ .

But,  $u \rightarrow 0$  exponentially fast. So e.g.



$$d = \text{"Penetration depth"} = 1/K = \frac{\hbar}{\sqrt{2m(V-E)}} \left\{ \begin{array}{l} \text{if } V \gg E, \\ \text{rapid decay!} \end{array} \right.$$

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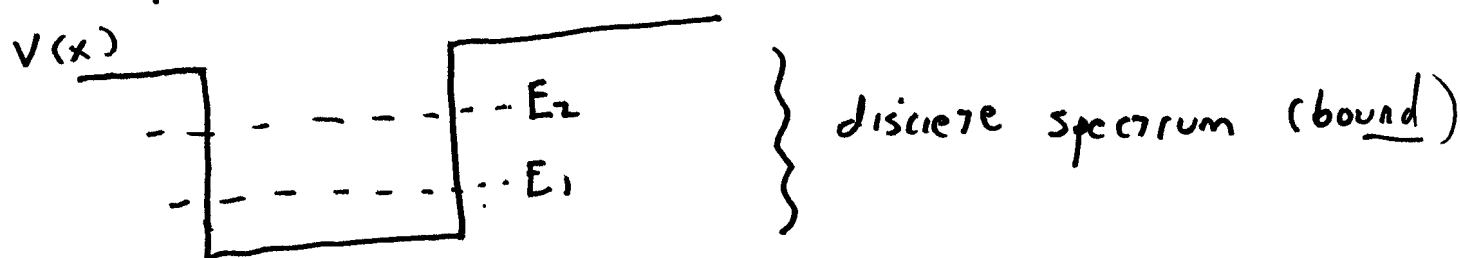
With these ideas, you can guess / sketch  $U(x)$  in many cases:

$u, u'$  continuous always (if  $V, E$  finite)

$E > V \Rightarrow$  sinusoidal. Bigger  $E - V \Rightarrow$  smaller  $\lambda$   
+ (usually) smaller  $|\psi|$

$E < V \Rightarrow$  decaying exponential. Bigger  $V - E \Rightarrow$  faster decay

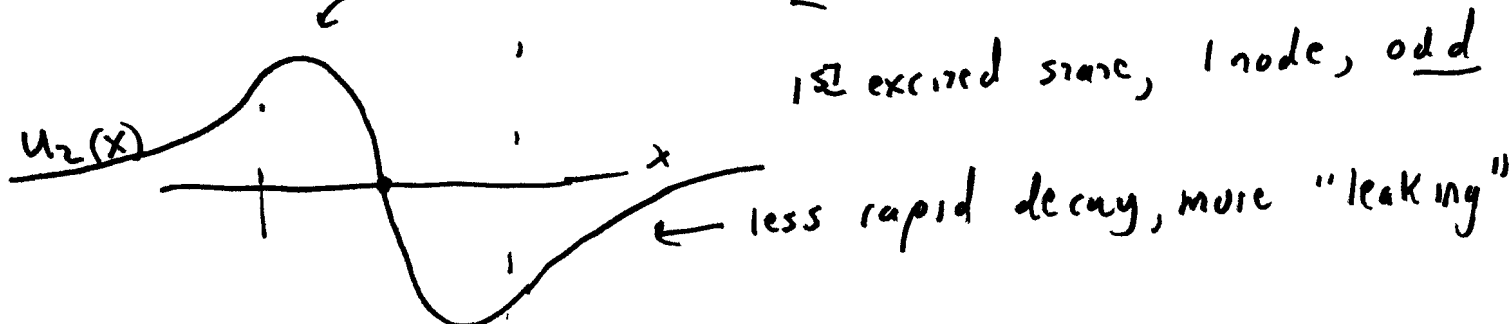
Example: Finite square well } continuous spectrum (free)



Ground state, even about center

[Turns out, you can prove, if  $V(x)$  is even,  $V(x) = V(-x)$ , then  $u(x)$  must be even or odd, so  $|u(x)|^2$  is symmetric!]

sinusoidal



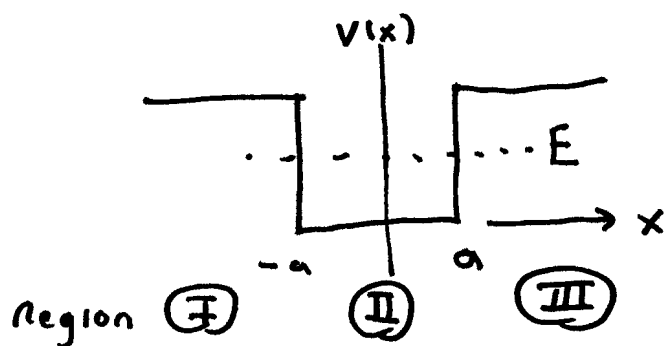
1st excited state, 1 node, odd

Let's work this out more rigorously to check, ok?

Let  $V(x) = \begin{cases} 0 & -a < x < a \\ +V_0 & \text{outside} \end{cases}$  (Note: I shifted the zero w.r.t. Griffiths)

Assume  $0 < E < V_0$ , i.e. bound state

Let's start with even sol's for  $u(x)$ :



We've solved for  $u(x)$  in  
Each region - already

Region (I)  $u(x) = C e^{\kappa x} + \dots / e^{-\kappa x}$ , with  $\kappa \equiv \sqrt{2m(V_0 - E)} / \hbar$   
No good, blows up

Region (II)  $u(x) = A \cos \kappa x + B \sin \kappa x$ , with  $\kappa = \sqrt{2mE} / \hbar$   
No good, looking for even sol'n right now

Region (III)  $u(x) = C e^{-\kappa x}$

→ same C, 'cause looking for even sol'n!

Boundary conditions:  $u_{\text{III}}(x)|_{x=+a} = u_{\text{II}}(x)|_{x=a}$

$u'_{\text{III}}(x)|_{x=+a} = u'_{\text{II}}(x)|_{x=a}$

(Conditions at  $x = -a$  add no new info, since it's even!)



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we have 3 unknowns:  $E$ ,  $A$ , and  $C$  → coefficients  
 $\hookrightarrow$  the energy eigenvalue.

we have 2 Boundary conditions + Normalization, so we can solve for 3 unknowns!

Continuity of  $u$  says  $Ce^{-\kappa a} = A \cos Ka$

" "  $u'$  says  $-\kappa C e^{-\kappa a} = -AK \sin Ka$

Dividing these  $\Rightarrow -\kappa = -K \tan Ka$ .

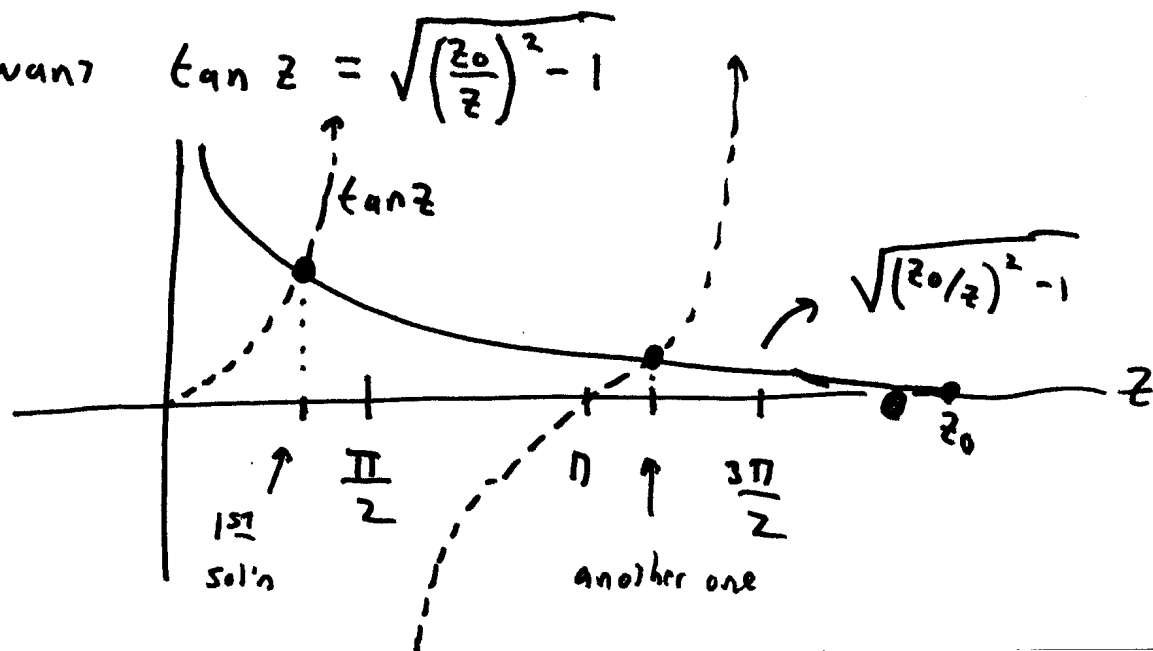
Nasty, transcendental eq'n with one variable,  $E$ .

$$\text{or } \tan\left(\sqrt{\frac{2mE}{\hbar^2}} a\right) = \frac{\kappa}{K} = \sqrt{\frac{V_0 - E}{E}}$$

Can solve numerically, or graphically,

$$\left. \begin{aligned} \text{Let } z \equiv Ka &= \sqrt{\frac{2mE}{\hbar^2}} a \\ \text{Let } z_0 &= \sqrt{\frac{2mV_0}{\hbar^2}} a \end{aligned} \right\} \text{ so } \frac{V_0}{E} = \left(\frac{z_0}{z}\right)^2$$

So we want  $\tan z = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$



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I see there are sol's for  $z \Rightarrow E$

They are discrete. There is a finite # of them.

No matter what  $z_0$  is, there's always at least one.

If  $V_0 \rightarrow \infty$ , sol's are near  $\frac{\pi}{2}, \frac{3\pi}{2}, \dots$

$$\text{giving } E = \frac{\hbar^2}{2ma^2} z^2 = \frac{\hbar^2 \pi^2}{2m(2a)^2} (1^2, \text{ or } 3^2, \dots)$$

which is exactly the even sol's for our  $\infty$  well with width 2a

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[ Knowing  $z \Rightarrow E \Rightarrow K$  and  $\mathcal{K}$ , our B.C.'s give  
a in terms of  $C$ , and normalization fixes  $C$ ... ]

I leave the odd sol's to you. The story is similar,  
+ you will get, as your transcendental eq'n,

$$\cot(ka) = -\frac{\mathcal{K}}{K}$$

This time, it's not always the case there's even 1 sol'n.  
(The well needs to be deep enough to get a 1st excited state,  
but even a shallow well has one bound state)

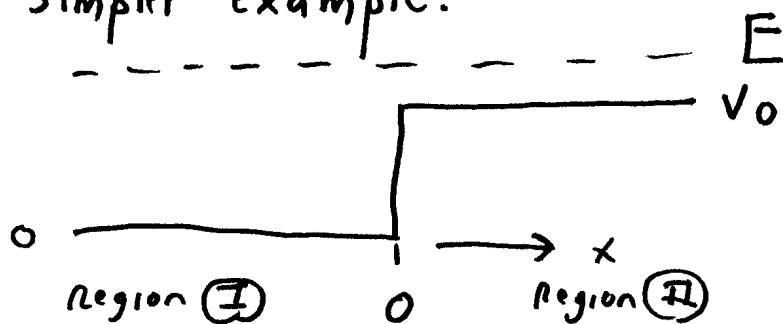
STP QM 3220 2.45

What about  $E > V_0$ ? This is the continuum region, also called "Scattering States", because  $U$  will be nonzero off to  $\infty$ , it's like a particle comes in, interacts with our potential well, + is affected (scattered)

~~the~~ Loss of physics in this example (- like "shooting electrons" at some object it interacts with.)

Griffiths treats the square well, I'll do a

simpler example:



The STEP potential.  
(think of wire, w. junction:  
Higher Voltage on right,  
lower " " left)

We could build a wave packet, send it in from left with  $E > V_0$ , + solve for time dependence. That's the right way to make this physically realistic. But our approach will be simpler. Look for  $U(x)$  which solves S.E. and which is physically interpreted as "something coming in from the left".

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$$u_I(x) = A e^{iKx} + B e^{-iKx} \quad \text{with } K = \sqrt{\frac{2mE}{\hbar^2}}$$

$$u_{II}(x) = C e^{iK'x} + D e^{-iK'x} \quad \text{with } K' = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

Remember our interpretation of free particle solns from awhile back, the  $A e^{iKx}$  term  $\Rightarrow$  incoming wave, traveling rightwards with  $p = \hbar K$

$B e^{-iKx}$  represents a left moving wave, but it's in region I, so it must physically represent reflected wave (in steady state, remember!)

$C e^{iK'x}$  represents "transmitted wave", going right in region II.

We set  $D = 0$ , because otherwise we'd have "incoming wave from the right", and our physics condition was "waves coming in from left."

Boundary conditions:  $u_I(x) \big|_{x=0} = u_{II}(x) \big|_{x=0}$   
 $u_I'(x) \big|_{x=0} = u_{II}'(x) \big|_{x=0}$

What about normalization? These are not normalizable  $u$ 's, (like the free particle, you must make a packet in the end)

What are we after? Well, given  $A$  (= amplitude entering)  
 I want to know  $T$ , "TRANSMISSION ~~coefficient~~" and  
 $R$ , "Reflection Coefficient", how much of an incident wave  
 moves on, + how much "bounces"?

Intuitively,  $\frac{B}{A}$  will tell us about  $R$   
 and  $\frac{C}{A}$  " " " "  $T$

But we need to be careful ...

Let's first compute these though.

B.C. on  $u$  gives  $A + B = C$

(Because  $e^{\pm i k \cdot 0} = 1$ )

B.C. on  $u'$  gives  $K(A - B) = K' C$

Divide by  $K$ , add  $\Rightarrow$

$$\text{so: } 2A = C \left(1 + \frac{K'}{K}\right) \text{ or } \frac{C}{A} = \frac{2}{1 + K'/K}$$

$$\text{put this back into 1st eq'n: } A + B = C = \frac{2}{1 + K'/K} A$$

$$\Rightarrow \frac{B}{A} = \frac{2}{1 + K'/K} - 1 = \frac{1 - K'/K}{1 + K'/K}$$

$$\text{with } K'/K = \sqrt{\frac{E - V_0}{E}}$$

So now we can talk about interpreting these coefficients  
 of (non-normalizable) "plane waves".

Probability Current  $J(x, t)$ .

$$\text{Prob}(a < x < b) \stackrel{\text{measuring } x}{=} \int_a^b |\Psi(x, t)|^2 dx$$

$|\Psi|^2$  evolves with time, smoothly. So probability "flows" in (and out) of this region!

I claim  $\frac{\partial |\Psi(x, t)|^2}{\partial t} = - \frac{\partial}{\partial x} (\underbrace{J(x, t)}_{\text{"Probability current" } *})$

where  $J(x, t) = \frac{i\hbar}{2m} \left( \Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x} \right)$  you need to do the algebra in prev. eq'n!

so  $\frac{\partial P}{\partial t} + \frac{\partial J}{\partial x} = 0$  where  $P = \text{Prob density} = |\Psi|^2$

or, integrating  $a \rightarrow b$

$$\frac{\partial}{\partial t} \int_a^b P \cdot dx = - \int_a^b \frac{\partial J}{\partial x} dx = -(J(b) - J(a)) = J(a) - J(b)$$

rate of "buildup" in  $(a, b)$  region = "current in at  $a$ " - "current out at  $b$ "

\* Just like electric charge, the name "current" is good!

Note that  $J(x, t) = \frac{i\hbar}{2m} (-2i) \text{Im} \left( \Psi^* \frac{\partial \Psi}{\partial x} \right) = \frac{\hbar}{m} \text{Im} \left( \Psi^* \frac{\partial \Psi}{\partial x} \right)$

For plane wave  $\Psi = A e^{i(kx - \omega t)}$

$$\frac{\partial \Psi}{\partial x} = iK \Psi, \text{ so } J = \frac{\hbar}{m} \text{Im} (iK \Psi^* \Psi)$$

$$\boxed{J = \frac{\hbar}{m} |A|^2 K} \quad \text{Plane waves}$$

This is key:  $\frac{\hbar K}{m}$  looks like  $\frac{p}{m} = v$

So "flow of probability" in a plane wave is like  $v \cdot |A|^2$

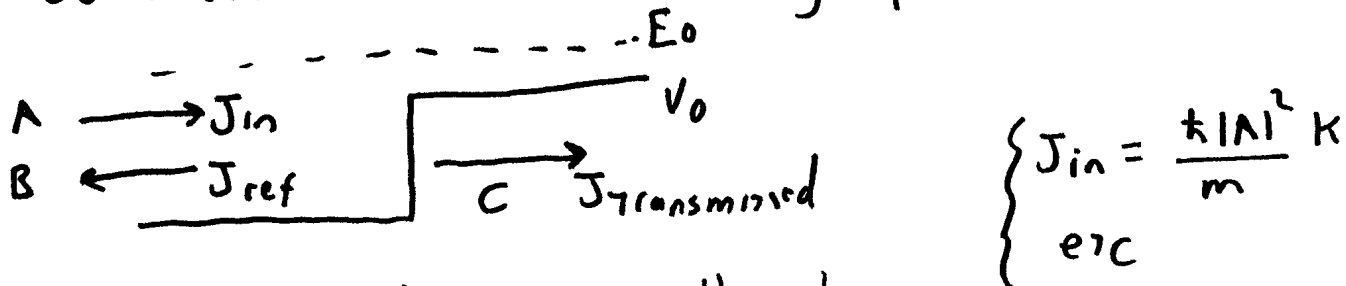
$\uparrow$  fast movement  $\Rightarrow$  lots of flow  
 $\uparrow$  large amp  $\Rightarrow$  lots of flow

Think of electric current, where  $j = \rho v$

$\uparrow$  speed of charges  
 $\uparrow$  charge density

Very similar!  $|A|^2$  plays the role of  $\rho$

So back to our Scattering problem:



If  $J =$  "probability current", then

$$R = \text{Reflection coefficient} = \frac{J_{refl}}{J_{inc}} \left( = \frac{|B|^2}{|A|^2} \right)$$

$$T = \text{Transmission coeff} = \frac{J_{trans}}{J_{inc}} \left( = \frac{|C|^2}{|A|^2} \frac{K'}{K} \right) \quad \left( \begin{array}{l} \text{The } K'/K \\ \text{is surprising} \\ \text{but makes} \\ \text{sense...} \end{array} \right)$$

we solved for  $B/\Lambda$  and  $C/\Lambda$ , so here

$$R = \left( \frac{1 - K'/K}{1 + K'/K} \right)^2$$

$$T = \frac{4}{(1 + K'/K)^2} \cdot \frac{K'}{K}$$

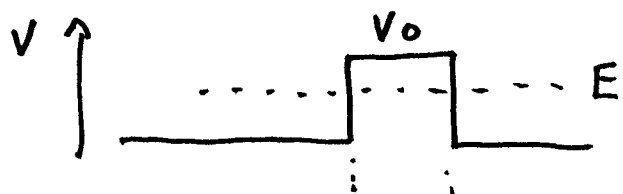
} convince yourself  
 $R + T = 1$  !

If  $V_0 \rightarrow 0$ ,  $\frac{K'}{K} = \frac{E - V_0}{E} \rightarrow 1$ ,  $\begin{cases} R \rightarrow 0 \\ T \rightarrow 1 \end{cases}$  makes sense!  
"No barrier"

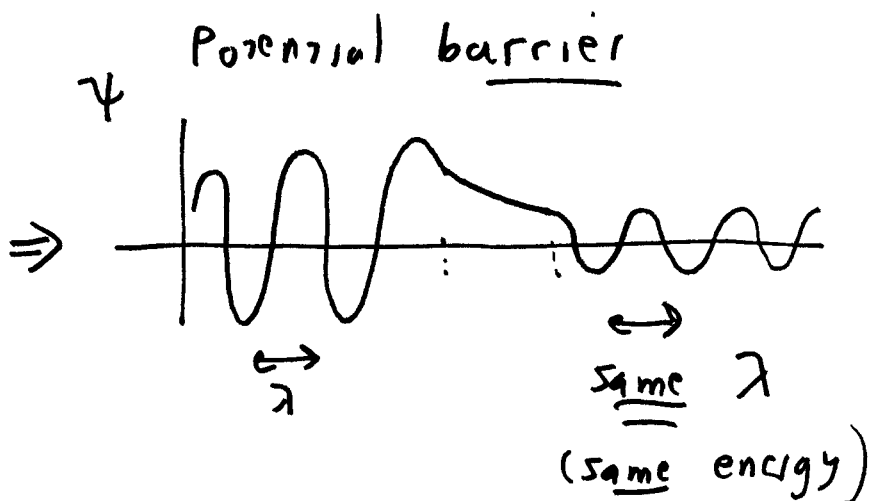
If  $E \rightarrow V_0$ ,  $\frac{K'}{K} \rightarrow 0$   $\begin{cases} R \rightarrow 1 \\ T \rightarrow 0 \end{cases}$  makes sense!  
"Forbidden" to go past barrier!

(Classically, if  $E > V_0$ ,  $R = 0$ . QM gives something new here)

other examples:



Pain, but  
Can compute  $R, T$ .



$E < V_0$ :  $T$  is not 0, even though classically it would be.

This is Tunneling.

$E > V_0$ : classically,  $T = 1$ , but QM  $\Rightarrow$  some reflection